Universidade Federal de Uberlândia
Engenharia Eletrônica e de Telecomunicações

- Processamento digital de sinais –
Capítulo 6 – Transformada z

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A transformada z

- O que é uma transformada ?

- Domínio z (plano z complexo)
  - caracterização, estabilidade

- Definição:

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Transformada inversa

$$x(n) \leftrightarrow X(z)$$

- Região de convergência (ROC ou RDC)
  - Valores de z para os quais X(z) tem valor finito

- Exemplo: calcule a transformada z dos sinais

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 7, 0, 1\}$$

$$x_2(n) = \{1, 2, \underset{\uparrow}{3}, 7, 0, 1\}$$

$$x_3(n) = \delta(n)$$

$$x_4(n) = \delta(n-k)$$

$$x_5(n) = \left(\frac{1}{2}\right)^n u(n)$$

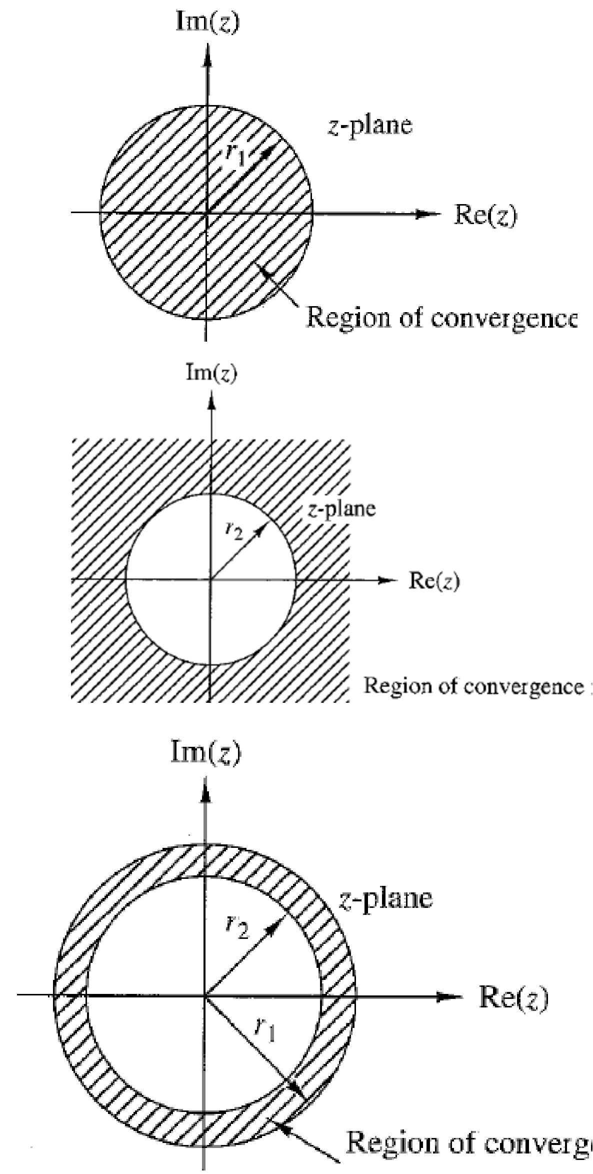
- Considere z como: $z = re^{j\theta}$

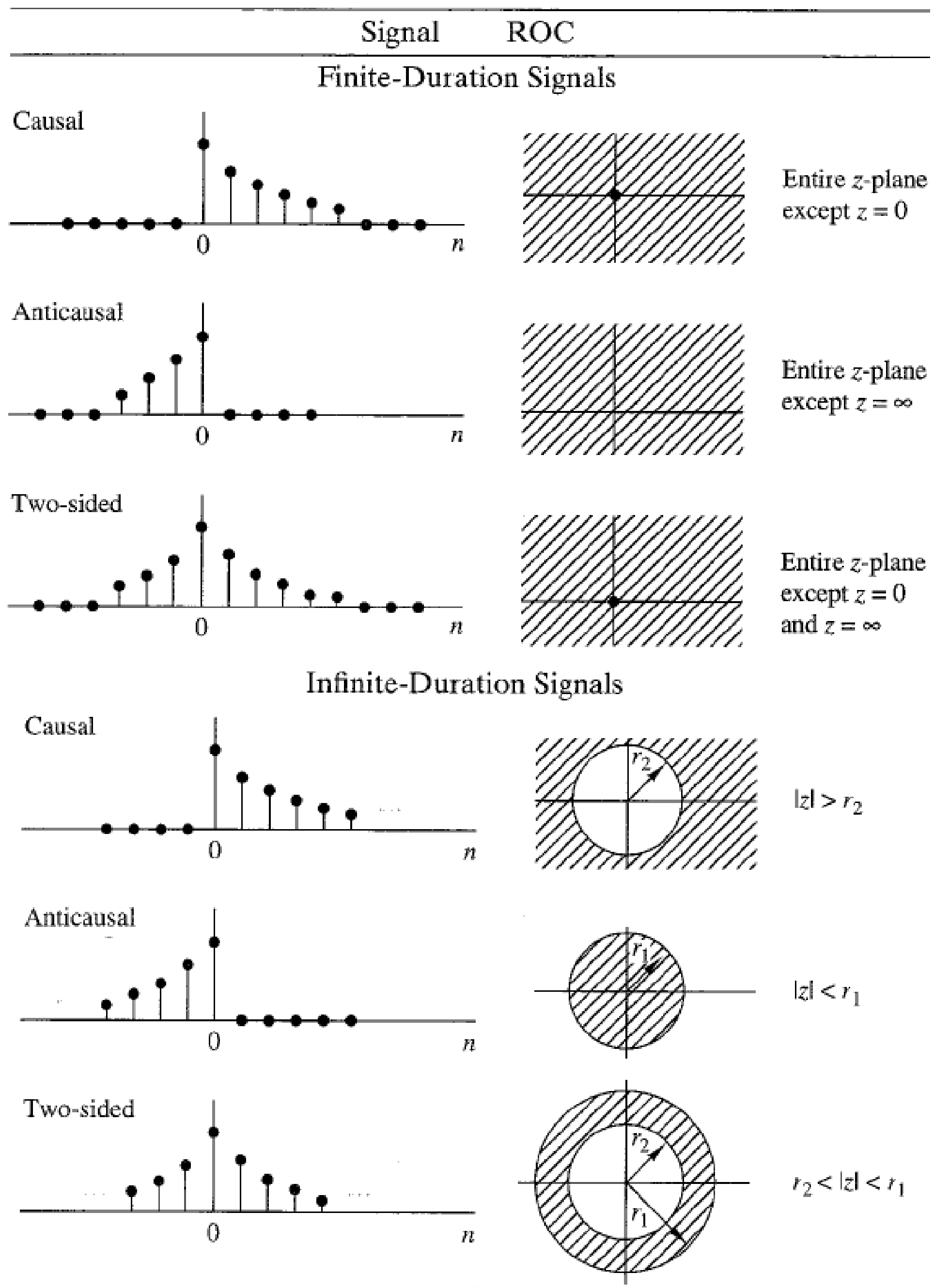
e $$X(z)\Big|_{z=re^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n}$$

e $$|X(z)| = \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right|$$

$$= \sum_{n=-\infty}^{-1} \left| x(n) r^{-n} \right| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$

$$= \sum_{n=1}^{\infty} \left| x(-n) r^{n} \right| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$





Signal
ROC
Finite-Duration Signals
Causal
Entire z-plane except z = 0
Anticausal
Entire z-plane except z = ∞
Two-sided
Entire z-plane except z = 0 and z = ∞
Infinite-Duration Signals
Causal
|z| > r2
Anticausal
|z| < r1
Two-sided
r2 < |z| < r1

- Propriedades da transformada z

P.1) Linearidade

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \leftrightarrow X(z) = a_1 X_1(z) + a_2 X_2(z)$$

Exemplo: determine a transformada z do sinal

$$x(n) = [3(2^n) - 4(3^n)]u(n)$$

P.2) Deslocamento no tempo

$$x(n) \leftrightarrow X(z)$$

$$x(n-k) \leftrightarrow z^{-k}X(z)$$

P.3) Escala no domínio z

$$x(n) \leftrightarrow X(z) \qquad ROC: r_1 < |z| < r_2$$

$$a^n x(n) \leftrightarrow X(a^{-1}z) \qquad ROC: |a| r_1 < |z| < |a| r_2$$

P.4) Diferenciação no domínio z

$$x(n) \leftrightarrow X(z)$$

$$nx(n) \leftrightarrow -z\frac{dX(z)}{dz}$$

P.5) Convolução de duas sequências

$$x_1(n) \leftrightarrow X_1(z)$$

$$x_2(n) \leftrightarrow X_2(z)$$

$$x(n) = x_1(n) * x_2(n) \leftrightarrow X(z) = X_1(z)X_2(z)$$

$$x(n) = Z^{-1}\{X(z)\}$$

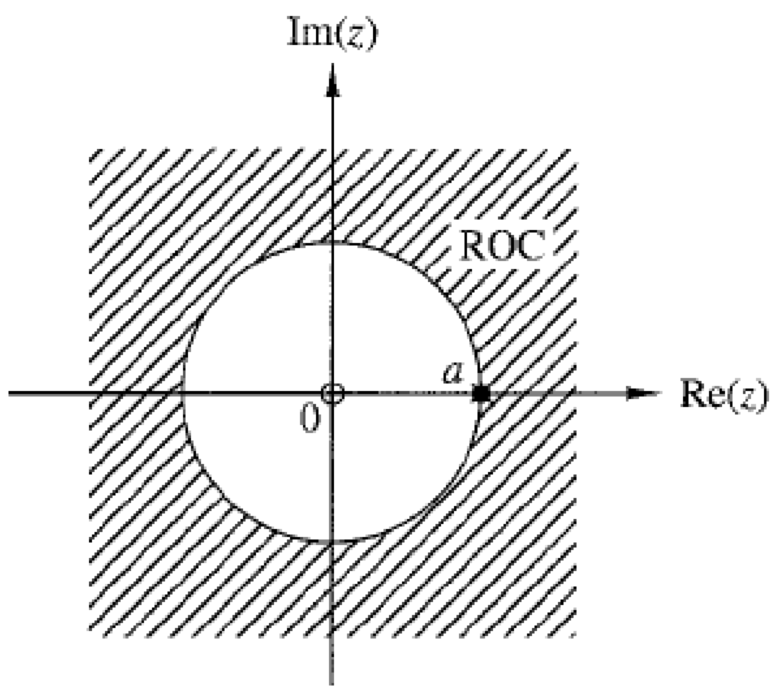
Property	Time Domain	z-Domain	ROC
Notation	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z=0$ if $k>0$ and $z=\infty$ if $k<0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\mathrm{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\mathrm{Im}\{x(n)\}$	$\frac{1}{2}j[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}\,dv$	At least, $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

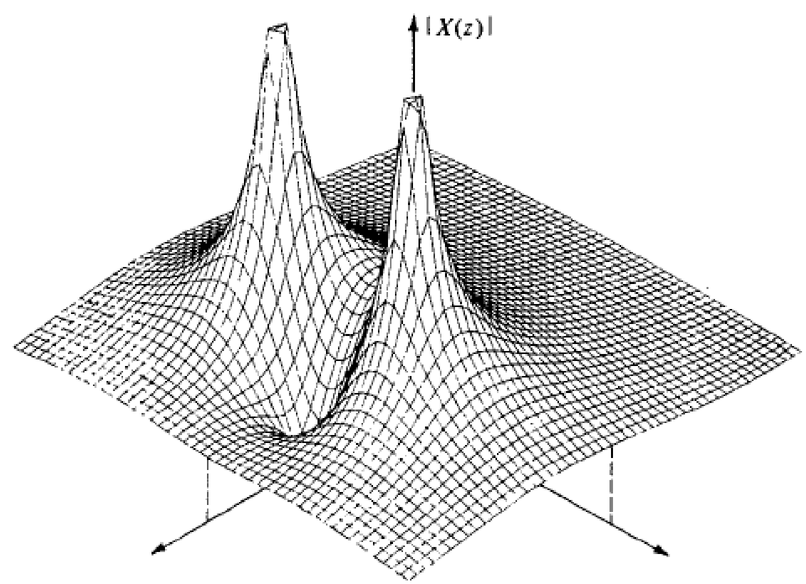
	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n\sin\omega_0 n)u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

- Polos e zeros

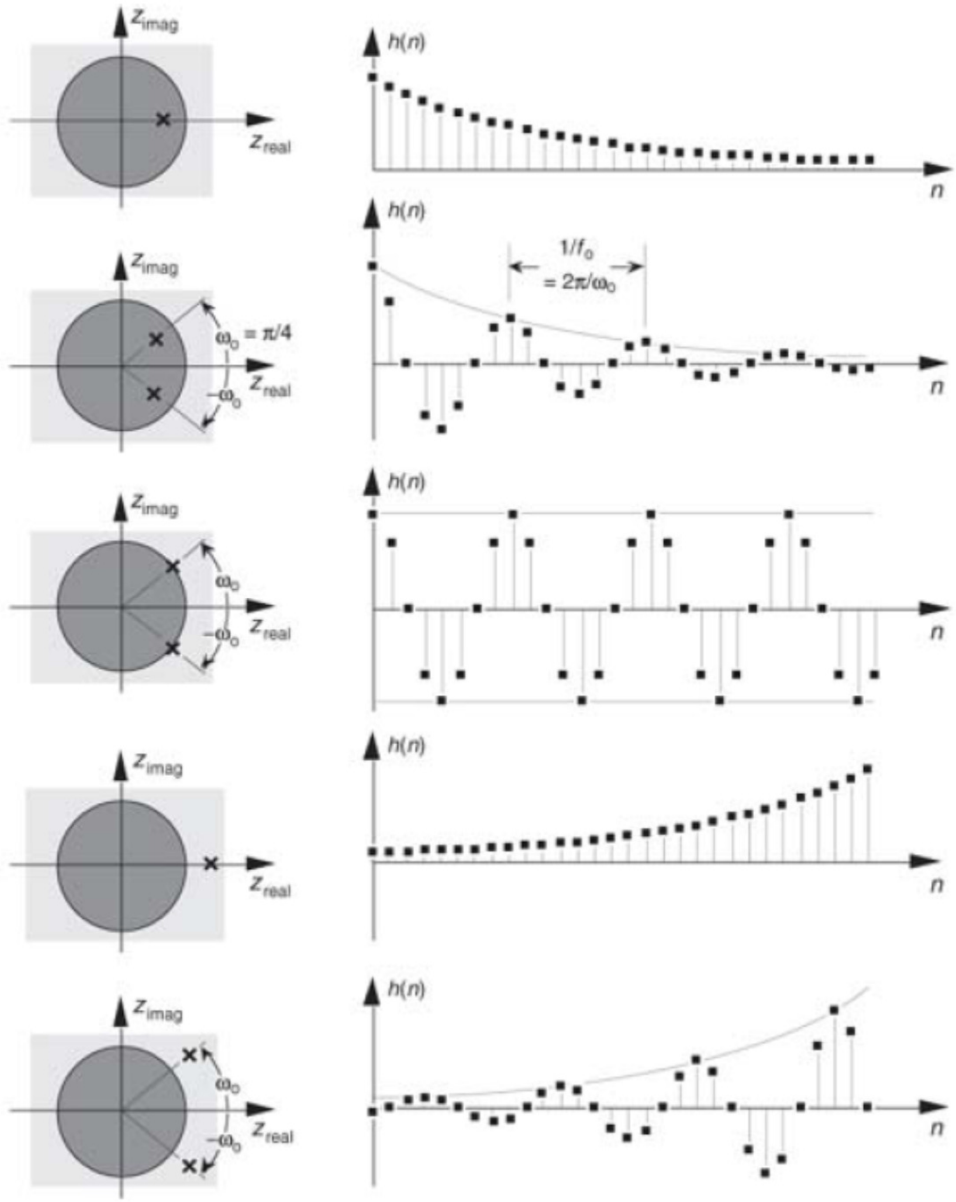
$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \dots b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots a_M z^{-M}}$$

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2)\dots(z - z_M)}{(z - p_1)(z - p_2)\dots(z - p_N)}$$



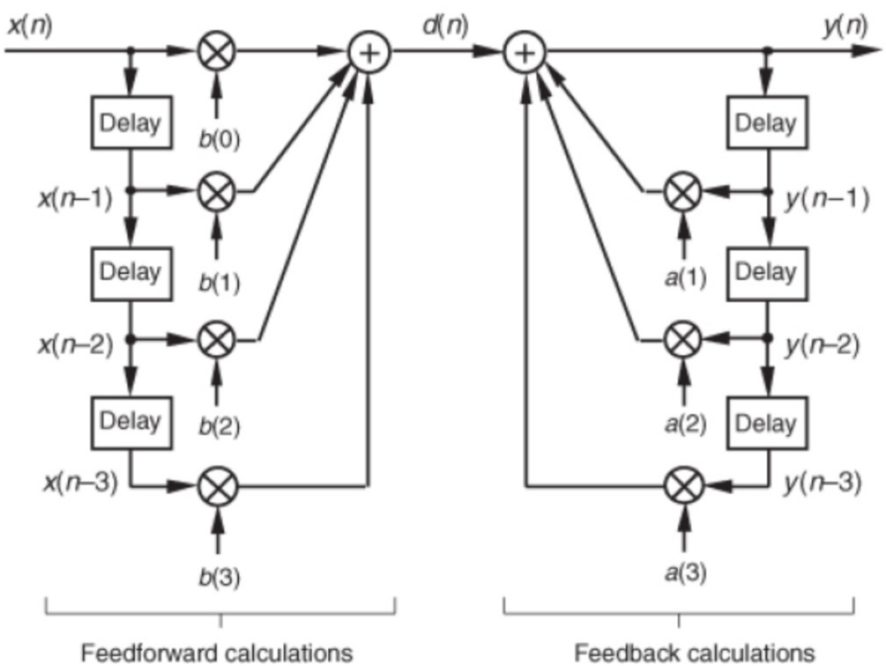


- Comportamento dos sinais



Análise de filtros IIR por Z

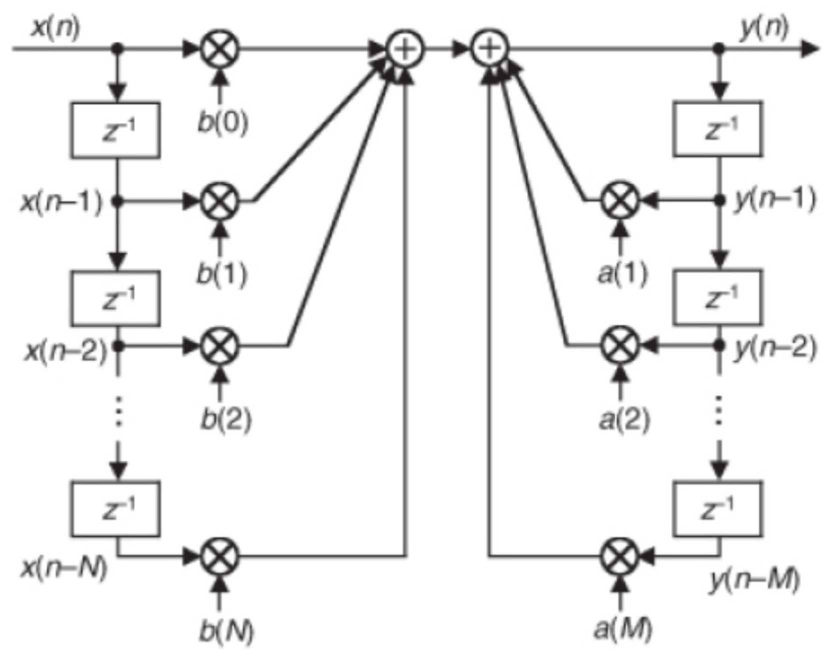
- Modelo:



(6-2)

$$y(n) = b(0)x(n) + b(1)x(n-1) + b(2)x(n-2) + b(3)x(n-3)$$

$$+ a(1)y(n-1) + a(2)y(n-2) + a(3)y(n-3).$$



$$Y(z) = b(0)X(z) + b(1)X(z)z^{-1} + b(2)X(z)z^{-2} + \ldots + b(N)X(z)z^{-N}$$

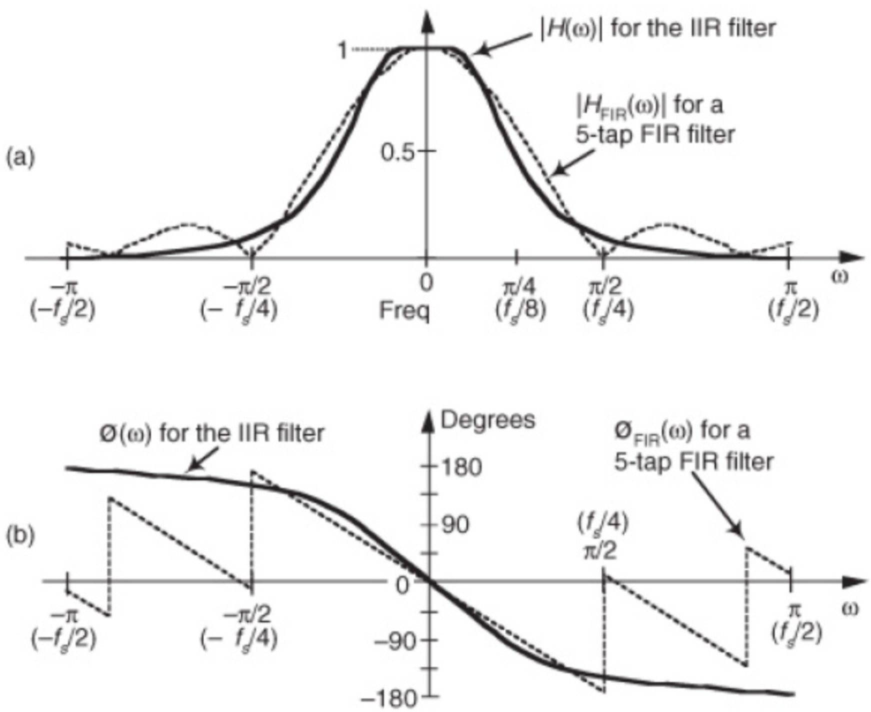
$$+ a(1)Y(z)z^{-1} + a(2)Y(z)z^{-2} + \ldots + a(M)Y(z)z^{-M}$$

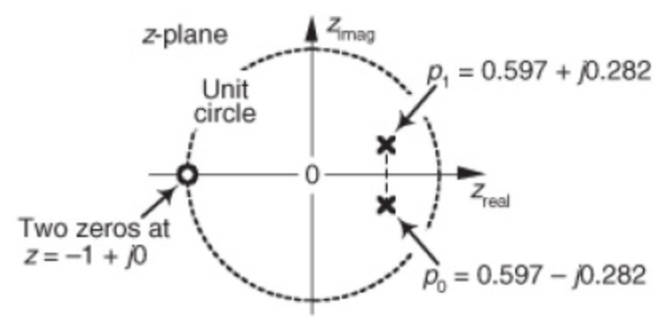
- Eq. Geral:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{1 - \sum_{k=1}^{M} a(k)z^{-k}}.$$

- Exemplo 1:

$$y(n) = 0.0605 \cdot x(n) + 0.121 \cdot x(n-1) + 0.0605 \cdot x(n-2)$$
$$+ 1.194 \cdot y(n-1) - 0.436 \cdot (n-2).$$





... Fatorando:

$$H(z) = \frac{(z-z_0)(z-z_1)}{(z-p_0)(z-p_1)} = \frac{(z+1)(z+1)}{(z-0.597+j0.282)(z-0.597-j0.282)}.$$